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Dendritic structures for fluid flow: Laminar, turbulent and constructal design

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Abstract

Fluid flow in dendritic structures is approached based on hydrodynamics and a geometric description of the network. The hydrodynamic performance of the network, composed of series of rough ducts, is studied for both laminar and turbulent flow regimes. Transient response of internal fluid pressure is also modelled and analyzed. The oscillatory character of the internal pressure is linked with characteristics of the fluid and characteristics of the network.

Keywords: Dendritic flow structures; Tree-shaped flow structures; Laminar; Turbulent; Oscillatory flow; Constructal design

1. Introduction

Dendritic (tree-shaped) flow structures play a basic role in animate and inanimate systems (Bejan, 2000; McCulloh et al., 2003; Reis et al., 2004). The study of these structures, though dating back to the early 20th century (Hess, 1914; Murray, 1926), is still of great interest due to their widespread applications in engineering, geophysics and physiology. During the first half of the twentieth century, Hess (1914) and Murray (1926) analyzed the distribution of blood vessel sizes of the circulatory system. The relationship known in physiology as the Hess–Murray law describes the optimal ratios of duct diameters in a bifurcation (Bejan, 2000; McCulloh et al., 2003). Bejan and co-authors Bejan (2000, 2005) and Bejan and Lorente (2006, 2007, 2008), based on the constructal law, theoretically investigated fluid flow in networks by minimizing the hydraulic resistance with the network volume constrained, and they obtained similar relationships to those reported in the literature (Hess, 1914; Murray, 1926). Besides, they showed that the best flow path that makes the connection of one point with an infinity of points (line, area or volume) is a network bifurcating on several levels. In summary, dendritic networks are the constructal design for providing easier flow access between points, areas and volumes. The evolution of these flow networks is not toward compactness or complexity but toward maximization of flow access, and is deducible from principle (i.e., the constructal law).

A study based on the constructal law derived the main features of lung architecture. Reis et al. (2004) showed that a bifurcating series of ducts, with m bifurcations connected to $2m$ alveoli, is the best lung structure. For humans, they found $m = 23$, which matches closely the actual number of bifurcations in the human lung. Besides, they predicted that the ratio of the square of the airway diameter to its length should be constant within a species and is related to the characteristics of the space allocated to the respiratory process. The interest in dendritic-shaped flow structures is

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spreading through different fields. Dendritic networks for cooling (Wang et al., 2006), applications for single-phase flow and two-phase flow (Bejan, 2002; Senn and Poulikakos, 2004, Kwak et al., 2009), dendritic-shaped heat and mass exchangers (Tondeur et al., 2000) and microvascular lab-on-a-chip systems (Lim et al., 2003; Emerson et al., 2006) have been proposed in the literature. The multi-scale structure of a dendritic flow network was also investigated by Queiros-Conde et al. (2007). Miguel (2008) studied the pressure in capillary dendritic networks, based on thermodynamics and geometric description of the network. The development of embedded dendritic vasculatures for smart materials with volumetric functionalities, such as self-healing and self-cooling, has recently been proposed (Wang et al., 2007; Lee et al., 2008; Bejan and Lorente, 2008).

In this paper, we study the fluid flow in dendritic networks. We derive analytical expressions for steady and transient flows, both for laminar and turbulent conditions. An approach is also proposed to describe the transient response of internal fluid pressure within dendritic structures with deforming walls and transporting fluids with different properties.

2. Dendritic flow structures: geometric and operational parameters

The geometric parameters of a dendritic flow structure are defined in Fig. 1. This network has N branches of ducts, from level 0 to level n. Each duct branches into m daughter branches at the next level. The ducts are cylindrical structures of diameter D_i, and lengths L_i, $i = 0, 1, \ldots, n$. The diameter and length of these ducts are sized relative to one another, in accordance with $D_{i+1}/D_i = a_D$ and $L_{i+1}/L_i = a_L$, where a_D and a_L are scale factors independent of i (Bejan, 2000; McCulloh et al., 2003; Reis et al., 2004). The relationship between the size of the first duct (level 0) and the size of ducts at level i is given by

$$\frac{D_i}{D_0} = a_D^i \quad \text{and} \quad \frac{L_i}{L_0} = a_L^i. \tag{1}$$

One basic feature of dendritic structures is that pairing, or bifurcation of ducts (dichotomy), is the constructal design of providing effective flow access (Bejan, 2000; Bejan and Lorente, 2006, 2008). Therefore, m takes a value of 2 (Fig. 1). If the flow is laminar, the minimization of flow resistance yields the scale factor $a_D = 2^{-1/3}$, which in physiology is known as Hess–Murray law. Moreover, for bifurcations ($m = 2$) the scale factors a_L reported in the literature ranges from 2^{-1} to $2^{-1/3}$ (Bejan, 2000; Bejan and Lorente, 2008). According to Bejan and Lorente (2008) the optimal geometric ratios of duct lengths and diameters change in the same proportion and the scale factors are equal ($a_D = a_L = 2^{-1/3}$). Therefore, the geometric ratio D/L is preserved in going from each duct to its branch. If the flow is turbulent, the constructal law shows that the optimal scale factors a_D and a_L are $2^{-3/7}$ and $2^{-1/7}$, respectively (Bejan, 2000; Bejan and Lorente, 2008). In this situation the geometric ratio D/L^3 is preserved in going from each duct to its branch.

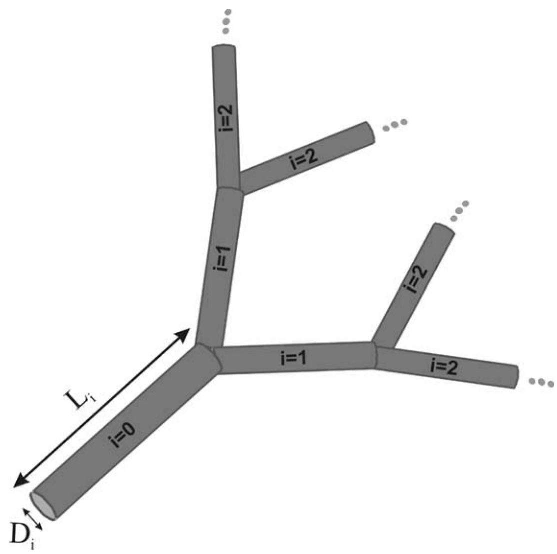

Fig. 1. Dendritic flow structure ($N = 3$; $m = 2$).

The effective length of the network, L, is $L=\sum_{i=0}^{n} L_i$ and the cross-sectional area, A, and the volume of the network, V, are, $A=\sum_{i=0}^{n} m^i \alpha_{sf} D_i^2$ and $V=\sum_{i=0}^{n} m^i \alpha_{sf} D_i^2 L_i$, respectively, where m is the number of branches, n the final branching level and α_{sf} is a shape factor (i.e., $\pi/4$). Therefore, we obtain

$$L=L_0\frac{1-a_L^{n+1}}{1-a_L},\quad A=\alpha_{sf}D_0^2\frac{1-(ma_D^2)^{n+1}}{1-ma_D^2},\quad V=\alpha_{sf}D_0^2L_0\frac{1-(ma_D^2a_L)^{n+1}}{1-ma_D^2a_L}. \tag{2}$$

These equations show the effective length, cross-sectional area and volume of the dendritic network in terms of geometric characteristics of the first duct, number of branches, branching level and scale factors.

3. Fluid flow through dendritic networks

For simplicity, we assume a one-dimensional fully developed flow, and inlet and outlet effects are neglected. The fluid flow, Q, can be related to the pressure difference, Δp, as

$$\rho\left(\frac{L}{A}\right)\frac{dQ}{dt}+\rho v\left(\frac{\lambda\ \text{Re}}{2}\right)\left(\frac{L}{D^2A}\right)Q=\Delta p. \tag{3}$$

Here t is the time, ρ the fluid density, v the kinematic fluid viscosity, Re the Reynolds number ($=QD/vA$) and λ the friction factor. Duct surfaces are rough to varying degrees. In the laminar region (Hagen–Poiseuille flow), roughness has a negligible effect and $\lambda=64/\text{Re}$. In the region so-called ''complete turbulence rough ducts'' or ''fully turbulent'', the friction factor is independent of Reynolds number and is a function of the relative roughness (Allen et al., 2005; Goldenfeld, 2006). Whenever $\text{Re}>900\varepsilon/\text{D}$, it turns out that

$$\lambda=\left[\frac{1}{1.74-2\log(2\varepsilon/D)}\right]^2,$$

where ε is the size of the roughness elements on the duct surfaces. For a laminar steady flow, substituting λ into Eq. (3) yields

$$\left(\frac{\Delta p}{\rho vQ}\right)_l=\frac{32L_0}{\alpha_{\text{sf}}D_0^4}\frac{1-ma_{D,l}^4a_{L,l}^{-1}}{1-(ma_{D,l}^4a_{L,l}^{-1})^{n+1}}. \tag{4}$$

In a similar procedure, the equation for turbulent flow is

$$\left(\frac{\Delta p}{\rho vQ}\right)_t=\frac{\text{Re}L_0}{2[1.74-2\log(2\chi/D_0)]\alpha_{\text{sf}}D_0^4}\frac{1-ma_{D,t}^5a_{L,t}^{-1}}{1-(ma_{D,t}^5a_{L,t}^{-1})^{n+1}}, \tag{5}$$

where $\text{Re}=Q/(v\alpha_{sf}D_0)$, $\chi=\varepsilon(1-a_{D,t})/(1-a_{D,t}^{n+1})$, and subscripts l and t mean laminar and turbulent regimes, respectively. Eqs. (4) and (5) define the resistance of the dendritic structure to fluid flow ($\Delta p/\rho vQ$). Considering that Δp, L_0 and the fluid properties (ρ,v) are constants, taking the logarithm on both sides of the each equation and differentiating yields

$$\left(\frac{\partial Q}{Q}\right)_{\Delta p,L_0,\rho,v}=4\left(\frac{\partial D_0}{D_0}\right)_{\Delta p,L_0,\rho,v}\text{ for laminar flow,}$$
$$\left(\frac{\partial Q}{Q}\right)_{\Delta p,L_0,\rho,v}=\frac{12.7-10\log(2\chi/D_0)}{3.48-4\log(2\chi/D_0)}\left(\frac{\partial D_0}{D_0}\right)_{\Delta p,L_0,\rho,v}\text{ for turbulent flow.} \tag{6}$$

If Q, L_0 and the fluid properties (ρ,v) are constants, then

$$\left(\frac{\partial(\Delta p)}{\Delta p}\right)_{Q,L_0,\rho,v}=-4\left(\frac{\partial D_0}{D_0}\right)_{Q,L_0,\rho,v}\text{ for laminar flow}$$

$$\left(\frac{\partial(\Delta p)}{\Delta p}\right)_{Q,L_0,\rho,v} = -\frac{12.7-10\log(2\chi/D_0)}{1.74-2\log(2\chi/D_0)}\left(\frac{\partial D_0}{D_0}\right)_{Q,L_0,\rho,v} \text{for turbulent flow} \tag{7}$$

Notice that in the laminar regime, when D_0 increases the fluid flow increases four times faster (Δp = const.) and the pressure difference decreases by a factor similar to that for the constant fluid flow. For turbulent flows, an increase of D_0 induces a decrease in pressure drop (Q = const.) that is two times greater than the increase of fluid flow (Δp = const.).

In order to seek a solution to transient flow (Eq. (3)), a realistic scenario is to consider a zero-flow initial condition (i.e., $Q=0$ at $t=0$). Therefore, we find that

$$Q_l = \frac{\alpha_{sf} D_0^4}{32 v\rho L_0}\frac{1-(ma_{D,l}^4 a_{L,l}^{-1})^{n+1}}{1-ma_{D,l}^4 a_{L,l}^{-1}}\Delta p\left\{1-\exp\left[-\frac{32}{D_0^2}\frac{(1-ma_{D,l}^4 a_{L,l}^{-1})(1-(ma_{D,l}^2 a_{L,l}^{-1})^{n+1})}{(1-ma_{D,l}^2 a_{L,l}^{-1})(1-(ma_{D,l}^4 a_{L,l}^{-1})^{n+1})}vt\right]\right\} \tag{8}$$

and

$$Q_t = \frac{2^{1/2}\alpha_{\mathrm{sf}}[1.74-2\log(2\chi/D_0)]D_0^{5/2}}{(\rho L_0)^{1/2}}\Delta p^{1/2}\left[\frac{1-(ma_{D,t}^5 a_{L,t}^{-1})^{n+1}}{1-ma_{D,t}^5 a_{L,t}^{-1}}\right]^{1/2}\frac{\exp(2\sigma t)-1}{1+\exp(2\sigma t)}, \tag{9}$$

with

$$\sigma = \frac{\Delta p^{1/2}}{(2\rho L_0 D_0)^{1/2}[1.74-2\log(2\chi/D_0)]}\left[\frac{1-ma_{D,t}^5 a_{L,t}^{-1}}{1-(ma_{D,t}^5 a_{L,t}^{-1})^{n+1}}\right]^{1/2}\frac{1-(ma_{D,t}^2 a_{L,t}^{-1})^{n+1}}{1-ma_{D,t}^2 a_{L,t}^{-1}}.$$

In conclusion, Eqs. (8) and (9), describe the transient fluid flow within dendritic structures under laminar and turbulent regimes, respectively.

4. Transient variation of pressure within the network

We now turn our attention to the pressure within the dendritic flow structure. The equation of mass conservation for fluid in the dendritic network is

$$\frac{d(V\rho)}{dt} = \rho Q \tag{10}$$

Consider that the wall of the ducts deforms to an extent that depends on the magnitude of pressure exerted to drive the fluid flow. For simplicity, we assume that the duct deformation can be obtained from $(V-V_0)/V_0 = \eta(p-p_0)$ where V_0 is the mean volume of the dendritic flow structure, p_0 the mean pressure and η the flexibility coefficient of the duct wall. If the fluid is considered to be a compressible medium, a state equation is required. For polytropic expansion or compression of ideal gas with constant heat capacity, the state equation can be written as $\rho = \rho_0(p/p_0)^{1/\beta}$, where β is the polytropic index ranging from 1 (isothermal) to 1.4 (isentropic gas expansion/compression). When the flow is laminar, substituting Eq. (10) into Eq. (3) yields

$$c_{1,l}\frac{d^2 p_l}{dt^2} + c_{2,l}\frac{dp_l}{dt}\frac{dp_l}{dt} + c_{3,l}\frac{dp_l}{dt} + p_l = p_0, \tag{11}$$

where

$$c_{1,l} = \rho L_0^2\left(\frac{1-ma_{D,l}^2 a_{L,l}^{-1}}{1-ma_{D,l}^2 a_{L,l}}\right)\left[\frac{1-(ma_{D,l}^2 a_{L,l})^{n+1}}{1-(ma_{D,l}^2 a_{L,l}^{-1})^{n+1}}\right]\left[\eta\left(1+\frac{1}{\beta}\right)+\frac{1-\eta p_0}{p_l}\right],$$

$$c_{2,l} = \rho L_0^2\left(\frac{1-ma_{D,l}^2 a_{L,l}^{-1}}{1-ma_{D,l}^2 a_{L,l}}\right)\left[\frac{1-(ma_{D,l}^2 a_{L,l})^{n+1}}{1-(ma_{D,l}^2 a_{L,l}^{-1})^{n+1}}\right]\left(\frac{\eta p_0-1}{p_l^2}\right),$$

$$c_{3,l} = \frac{32\rho L_0^2 v}{D_0^2}\left(\frac{1-ma_{D,l}^4 a_{L,l}^{-1}}{1-ma_{D,l}^2 a_{L,l}}\right)\left[\frac{1-(ma_{D,l}^2 a_{L,l})^{n+1}}{1-(ma_{D,l}^4 a_{L,l}^{-1})^{n+1}}\right]\left(\eta+\frac{1+\eta p_l-\eta p_0}{\beta p_l}\right).$$

Similarly, we conclude that the fluid pressure within a network of rough ducts in the turbulent flow regime is

$$c_{1,t}\frac{d^2 p_t}{dt^2} + c_{2,t}\frac{dp_t}{dt}\frac{dp_t}{dt} + p_t = p_0, \tag{12}$$

where

$$c_{1,t}=\rho L_0^2\left(\frac{1-ma_{D,t}^2a_{L,t}^{-1}}{1-ma_{D,t}^2a_{L,t}}\right)\left[\frac{1-(ma_{D,t}^2a_{L,t})^{n+1}}{1-(ma_{D,t}^2a_{L,t}^{-1})^{n+1}}\right]\left[\eta\left(1+\frac{1}{\beta}\right)+\frac{1-\eta p_0}{p_t}\right],$$

$$c_{2,t}=\rho L_0^2\left(\frac{1-ma_{D,t}^2a_{L,t}^{-1}}{1-ma_{D,t}^2a_{L,t}}\right)\left[\frac{1-(ma_{D,t}^2a_{L,t})^{n+1}}{1-(ma_{D,t}^2a_{L,t}^{-1})^{n+1}}\right]\left(\frac{\eta p_0-1}{p_t^2}\right)$$
$$+\frac{\rho L_0^3}{2c_t^2D_0}\left(\frac{1-ma_{D,t}^5a_{L,t}^{-1}}{1-ma_{D,t}^2a_{D,t}}\right)\left[\frac{1-(ma_{D,t}^2a_{L,t})^{n+1}}{1-(ma_{D,t}^5a_{L,t}^{-1})^{n+1}}\right]\left(\eta+\frac{1+\eta p_t-\eta p_0}{\beta p_t}\right).$$

and $c_t=[1.74-2\log(2\chi/D_0)]$.

Eqs. (11) and (12) describe the transient response of pressure variation within the network. The importance of these equations lies in the fact that both equations are similar to the one governing a damped harmonic oscillator. As the "damping factor" is greater than zero the system may or may not oscillate, depending on the relation between this factor and natural frequency. Eqs. (11) and (12) can not only be used to estimate the transient response of the internal pressure but also to obtain geometric parameters of the network if they are unknown.

5. Final remarks

Fluid flow has been the focus of many studies because of its importance in nature and in science. This study is focused on the understanding of fluid flow and fluid pressure in a dendritic flow network. An approach is presented by combining hydrodynamics with a geometric description of the network. First, we examined steady and transient fluid flows. Finally, an approach was presented whereby sudden transient response of internal pressure variation is shown to be related with the geometry and mechanical properties of the network, as well as the fluid properties.

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